Cholesky Decomposition and its importance in Quantitative Finance

Cholesky Decomposition plays a very important role in Quantitative Finance, especially in the Derivatives pricing part where we are dealing with multiple correlated assets. When we are trying to Model Products whose price/payoff is dependent on multiple assets, in many cases it’s preferable to use a Monte Carlo Simulation Approach rather than a closed form solution approach (A closed form solution may not be even feasible in many cases). When we deal with Monte Carlo simulation, in the basic form what we require is an array of Random Nos., in most cases it will be from the Standard Normal Distribution. But in case of Multiple Assets we need to generate Correlated Random Nos., and it’s for this purpose we use the Cholesky Decomposition, which acts as a filter which converts uncorrelated Random Nos. to Correlated random Nos. This is a proven method and we know it works, but we should not accept every concept blindly, we should try to understand why it works in the 1st place. Before we get into that let’s 1st see what is a Cholesky Decomposition?

Cholesky Decomposition

Definition: - **Cholesky decomposition** or **Cholesky factorization** is a decomposition of a Hermitian, positive-definite matrix into the product of a lower triangular matrix and its conjugate transpose.

Check the Determinant of your Correlation Matrix, which has to be positive for this decomposition to work.

References:-

https://en.wikipedia.org/wiki/Cholesky_decomposition

https://en.wikipedia.org/wiki/Positive-definite_matrix

https://en.wikipedia.org/wiki/Conjugate_transpose

For example we can say:-

\[ A = L \cdot L^T \]

Let A be a simple 2 X 2 Correlation Matrix, \( A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \)

\[ L = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.8660254 \end{bmatrix}, \quad L^T = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.8660254 \end{bmatrix} \]

You can verify the results yourself, that when you do a Matrix multiplication of \( L \cdot L^T \) you will get back A.
How does Cholesky Decomposition generate Correlated Random variables?

Let’s say you have a correlation Matrix “P”, which can be decomposed as \( P = L \times L^T \)

Let “X” be a vector of Un-correlated Random Nos.

By definition Correlated Random Nos. \( Z = L \times X \)

The Co-Variance Matrix of any Random Vector Y is given as \( E[YY^T] \)  

Let’s look at our new Random Vector Z

\[
E [ZZ^T] = E [(LX)(LX)^T] = E [LXX^T L^T] \text{ (By the rules of Matrix Algebra)}
\]

\[
E [ZZ^T] = L E[XX^T] L^T \text{ (As L and } L^T \text{ is a constant, it can come out of the Expectation)}
\]

\[
E[XX^T] = I \text{ (Please refer Notes for more details)}
\]

\[
E [ZZ^T] = L I L^T = L L^T = P
\]

**Hence the Random Vector Z has the desired correlation P.**
Notes
Let “X” and “Y” be 2 Random Vectors
Then $\text{Cov}(X, Y) = E[XY^T] - E[X](E[Y])^T$

When X, Y are from the Standard Normal Distribution, the above relation reduces to
$\text{Cov}(X, Y) = E[XY^T]$

When we want to calculate Co-Variance of “X” w.r.t. “X”, then that is nothing but the Variance of “X”.
$\text{Cov}(X, X) = \text{Var}(X) = I$ {I = Identity Matrix. As “X” is from the Standard Normal Distribution}